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Analytical theories of turbulence

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This paper surveys the current state of analytical attempts at a theory of turbulence. The formulation of the problem in terms of moments is discussed. The difficulty posed by the closure problem is examined in detail using the quasi-normal approximation as an example. The notion of dynamical relaxation by non-linear scrambling leads to the introduction of eddy relaxation times and the direct-interaction approximation. The properties of the direct-interaction approximation are indicated. Finally, a comparison is made between numerical solution of the equations of turbulence theory and direct numerical simulation of the Navier–Stokes equations.

1. Introduction

In the half century since G. I. Taylor introduced correlation functions (Taylor 1921), there have been various attempts to formulate a theory of turbulence using them. This paper critically reviews the current state of this theory.

The essential difficulty does not involve finding an exact mathematical description of turbulence but rather involves extracting useful information from a formally exact solution. For example, it will be seen in §2 that an infinite unclosed set of moment equations for correlation functions is obtained from the Navier–Stokes equations. This set of equations is formally exact but remains useless until an algorithm (called a closure scheme) reduces the problem to a finite set of integro-differential equations in an ordinary function space. The complete (unclosed) set of moment equations could be reformulated as a single functional-differential equation in a suitable Hilbert space, but this disguises the problem of extracting useful information. While it is commonly believed that finite-difference methods may usefully be applied to the solution of integro-differential equations for ordinary functions, such confidence is lacking for functional-differential equations or their formal solutions in terms of functional integrals. From a mathematical point of view, turbulence theory is embedded in the theory of approximation of functional-differential equations.

In this paper, we address the problem of computing the energy spectrum of homogeneous, isotropic, incompressible turbulence. Admittedly, this is a limited goal having little direct relevance to more practical turbulence questions such as Reynolds' problem of determining the pressure drop in turbulent pipe flow. Hopefully the next several years will see analytical theories applied to a

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variety of interesting shear flows, perhaps giving a satisfactory theoretical solution to problems such as pipe flow. The importance of this direction of turbulence research will be emphasized again at the conclusion of this paper.

Turbulence is homogeneous if the flow has identical statistical properties at all points in space; it is isotropic if the statistical properties are identical in all directions. Average properties of the velocity field will be interpreted as ensemble averages, that is, as averages over a large number of flow realizations prepared under nearly identical initial and boundary conditions. Each realization will be assumed to satisfy the Navier–Stokes equations,

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t).\nabla\mathbf{v}(\mathbf{x},t) = -\nabla p(\mathbf{x},t) + \nu\nabla^2\mathbf{v}(\mathbf{x},t), \tag{1.1}$$

$$\nabla.\mathbf{v}(\mathbf{x},t) = 0, \tag{1.2}$$

where $\mathbf{v}(\mathbf{x},t)$ is the velocity field, $p(\mathbf{x},t)$ the pressure, and ν the kinematic viscosity. The determination of the energy spectrum (defined by (2.14) below) is equivalent (by Fourier transformation) to the determination of the two-point correlation function $\langle \mathbf{v}(\mathbf{x},t)\,\mathbf{v}(\mathbf{x}',t)\rangle$ where $\langle\ \rangle$ denotes ensemble average; the mean velocity $\langle \mathbf{v}(\mathbf{x},t)\rangle$ may be assumed zero by homogeneity.

2. Moment equations and the quasi-normal approximation

It is natural to formulate the theory of homogeneous turbulence in Fourier space. A careful discussion of Fourier transforms requires more space than is available here so the necessary results will be quoted. Our purpose is to establish (2.10) and (2.11), whose form alone is necessary for the understanding of later sections. The interested reader may consult the paper of Orszag & Kruskal (1968) for more details.

Suppose that each realization of the ensemble is confined to a cube of side L with periodic boundary conditions; the limit $L\to\infty$ is taken later. In this finite cube each flow may be expanded in a Fourier series:

$$\mathbf{v}(\mathbf{x},t) = \sum \mathbf{u}(\mathbf{k},t)e^{i\mathbf{k}.\mathbf{x}}, \quad \mathbf{u}(0,t) = 0, \tag{2.1}$$

where periodicity requires

$$\mathbf{k} = \frac{2\pi}{L}\mathbf{n}, \quad \mathbf{n} = (n_1, n_2, n_3) \quad (n_i = 0, \pm 1, \pm 2, \ldots). \tag{2.2}$$

The transforms of (1.1), (1.2) are

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] u_\alpha(\mathbf{k},t) = -\tfrac{1}{2}iP_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\mathbf{p}} u_\beta(\mathbf{p},t)\, u_\gamma(\mathbf{k}-\mathbf{p},t), \tag{2.3}$$

$$k_\alpha u_\alpha(\mathbf{k},t) = 0, \tag{2.4}$$

$$P_{\alpha\beta\gamma}(\mathbf{k}) = k_\beta P_{\alpha\gamma}(\mathbf{k}) + k_\gamma P_{\alpha\beta}(\mathbf{k}), \tag{2.5}$$

$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta/k^2, \quad k = |\mathbf{k}|. \tag{2.6}$$

Greek subscripts range from 1 to 3; summation over repeated indices is implied.

In (2.3) the pressure has been eliminated in favour of the tensor $\mathbf{P}(\mathbf{k})$ by use of the incompressibility condition (2.4).

It may easily be shown that homogeneity requires that the only non-zero moments of the velocity field be of the form $\langle \mathbf{u}(\mathbf{k})\,\mathbf{u}(\mathbf{p})\ldots\mathbf{u}(\mathbf{q})\rangle$ with $\mathbf{k}+\mathbf{p}+\ldots+\mathbf{q}=0$. Consideration of the limit $L\to\infty$ shows the existence of

$$S_{\alpha\beta}(\mathbf{k},t)=\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3\langle u_\alpha(\mathbf{k},t)\,u_\beta(-\mathbf{k},t)\rangle, \tag{2.7}$$

$$T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},t)=\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^6\langle u_\alpha(\mathbf{k},t)\,u_\beta(\mathbf{p},t)\,u_\gamma(-\mathbf{k}-\mathbf{p},t)\rangle, \tag{2.8}$$

$$U_{\alpha\beta\gamma\delta}(\mathbf{k},\mathbf{p},\mathbf{q},t)=\lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^9\langle u_\alpha(\mathbf{k},t)\,u_\beta(\mathbf{p},t)\,u_\gamma(\mathbf{q},t)\,u_\delta(-\mathbf{k}-\mathbf{p}-\mathbf{q},t)\rangle, \tag{2.9}$$

where it is required that $\mathbf{k}\neq-\mathbf{p}\neq\mathbf{q}\neq-\mathbf{k}$ in (2.9). The tensors $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, etc., defined by (2.7)–(2.9), are called cumulants of the second, third, fourth order, respectively.

Equations for the cumulants are constructed by multiplying (2.3) by a suitable product of Fourier amplitudes and averaging. The first two equations obtained in this way are

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]S_{\alpha\beta}(\mathbf{k},t)=-\frac{i}{2}\int d^3p\,[P_{\alpha\rho\sigma}(\mathbf{k})\,T_{\beta\rho\sigma}(-\mathbf{k},\mathbf{p},t)+P_{\beta\rho\sigma}(-\mathbf{k})\,T_{\alpha\rho\sigma}(\mathbf{k},\mathbf{p},t)], \tag{2.10}$$

$$\begin{aligned}\left[\frac{\partial}{\partial t}+\nu(k^2+p^2+q^2)\right]T_{\alpha\beta\gamma}(\mathbf{k},\mathbf{p},t)&=-\frac{i}{2}\int d^3r\,[P_{\alpha\rho\sigma}(\mathbf{k})U_{\beta\gamma\rho\sigma}(\mathbf{p},\mathbf{q},\mathbf{r},t)\\&\quad+P_{\beta\rho\sigma}(\mathbf{p})U_{\alpha\gamma\rho\sigma}(\mathbf{k},\mathbf{q},\mathbf{r},t)+P_{\gamma\rho\sigma}(\mathbf{q})U_{\alpha\beta\rho\sigma}(\mathbf{k},\mathbf{p},\mathbf{r},t)]\\&\quad-i[P_{\alpha\rho\sigma}(\mathbf{k})\,S_{\beta\rho}(\mathbf{p},t)\,S_{\gamma\sigma}(\mathbf{q},t)+P_{\beta\rho\sigma}(\mathbf{p})\,S_{\alpha\rho}(\mathbf{k},t)\,S_{\gamma\sigma}(\mathbf{q},t)\\&\quad+P_{\gamma\rho\sigma}(\mathbf{q})\,S_{\alpha\rho}(\mathbf{k},t)\,S_{\beta\sigma}(\mathbf{p},t)],\end{aligned} \tag{2.11}$$

where $\mathbf{q}=-\mathbf{k}-\mathbf{p}$ in the latter equation. Equations for $\mathbf{U}$ and higher-order cumulants may similarly be obtained. The reader is urged not to be intimidated by the complexity of (2.10), (2.11), the details of which are not essential to an understanding of the subject.

There are two important features of (2.10), (2.11). First, it should be observed that (2.10) for $\mathbf{S}$ involves $\mathbf{T}$, while (2.11) for $\mathbf{T}$ involves $\mathbf{U}$. Similarly, the equation for $\mathbf{U}$ involves fifth-order cumulants and, in general, the equation for nth-order cumulants involves those of order $n+1$. The equations for cumulants form a coupled infinity of integro-differential equations called a cumulant hierarchy. It is evident that this hierarchy, though formally exact, does not solve the turbulence problem until an algorithm is given for extracting useful information from it. The dilemma posed by an infinite set of coupled equations is referred to as the closure problem.

The second interesting feature of the hierarchy concerns the terms of the form $\mathbf{SS}$ in (2.11). Together with similar terms in equations for higher-order cumulants this contribution distinguishes a formulation in terms of cumulants from one in

terms of moments. These terms arise from the kinematical effect that widely separated fluid elements in physical space should be statistically independent. This requires, for example, that, as $\mathbf{X} \to \infty$,

$$\begin{aligned}&\langle \mathbf{v}(\mathbf{x}_1, t_1)\, \mathbf{v}(\mathbf{x}_2, t_2)\, \mathbf{v}(\mathbf{x}_3 + \mathbf{X}, t_2)\, \mathbf{v}(\mathbf{x}_4 + \mathbf{X}, t_4)\rangle \\ &\quad \to \langle \mathbf{v}(\mathbf{x}_1, t_1)\, \mathbf{v}(\mathbf{x}_2, t_2)\rangle \langle \mathbf{v}(\mathbf{x}_3 + \mathbf{X}, t)\, \mathbf{v}(\mathbf{x}_4 + \mathbf{X}, t_4)\rangle \\ &\quad = \langle \mathbf{v}(\mathbf{x}_1, t_1)\, \mathbf{v}(\mathbf{x}_2, t_2)\rangle \langle \mathbf{v}(\mathbf{x}_3, t_3)\, \mathbf{v}(\mathbf{x}_4, t_4)\rangle\end{aligned}$$

for any fixed set of $\mathbf{x}_1, \ldots, \mathbf{x}_4, t_1, \ldots, t_4$. The terms **SS** account for this behaviour.

It remains to face the closure problem. In other physical problems (such as the kinetic theory of dilute gases) perturbation theory resolves the difficulty. The most evident small parameter at large Reynolds number is the viscosity; unfortunately, setting $\nu = 0$ does not resolve the closure difficulty and may compound it. More suitable expansion parameters on which to base a closure scheme have been sought without notable success. We will be more cavalier in our attempt to close the hierarchy; we shall obtain closure by dropping all effects of cumulants above some given order.

If the effects of **T** are neglected in the equation for **S** there results

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] S_{\alpha\beta}(\mathbf{k}, t) = 0, \tag{2.12}$$

whose solution

$$S_{\alpha\beta}(\mathbf{k}, t) = e^{-2\nu k^2 t} S_{\alpha\beta}(\mathbf{k}, 0) \tag{2.13}$$

includes only the effect of viscous dissipation. Non-linear transfer of energy is absent since $\mathbf{T} = 0$. This result gives a satisfactory low Reynolds number approximation, but energy transfer is expected to be important at high Reynolds numbers.

The second closure in this scheme, namely neglecting the effects of **U** in (2.11), is more suitable, since here **T** evolves to give non-zero energy transfer among Fourier modes. The closure obtained in this way is encouraged by the observation that **U** vanishes identically if the ensemble is Gaussian. Experiments indicate that many simple properties of turbulence are approximated by Gaussianity. Justification of the closure would follow from an expansion in some measure of non-Gaussianity that gives $\mathbf{U} = 0$ in lowest approximation. However, exact normality (Gaussianity) also requires $\mathbf{T} = 0$, which would give (2.12). For this reason the closure obtained by dropping fourth- and not third-order cumulants is called the quasi-normal approximation (Millionschtchikov 1941; Proudman & Reid 1954; Tatsumi 1957).

With two additional assumptions the equations of the quasi-normal approximation may be brought into standard form. First, we assume that the ensemble is exactly Gaussian at the initial instant $t = 0$. This initial state receives some justification on the basis of the maximal randomness principle (Kraichnan 1959); for our purposes it is justified as being as plausible an initial state as any other. Gaussianity at $t = 0$ requires $T_{\alpha\beta\gamma}(\mathbf{k}, \mathbf{p}, 0) = 0$. Non-zero values of **T** are developed in evolution according to (2.11) so that the flow is *not* Gaussian after the initial instant. The second assumption is isotropy, according to which

$$S_{\alpha\beta}(\mathbf{k}, t) = \tfrac{1}{2} U(k, t)\, P_{\alpha\beta}(\mathbf{k}), \tag{2.14}$$

where the scalar function $U(k,t)$ is related to the energy spectrum $E(k,t)$ by

$$E(k,t) = 2\pi k^2 U(k,t). \tag{2.15}$$

With these assumptions (2.10), (2.11) with $\mathsf{U} = 0$ become

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] U(k,t) = k^2 \int d^3p \int_0^t e^{-\nu(k^2+p^2+q^2)[t-s]}\, ds$$
$$\times [a(k,p,q)\, U(p,s)\, U(q,s) - b(k,p,q)\, U(k,s)\, U(q,s)], \tag{2.16}$$

with the geometrically determined coefficients $a(k,p,q)$, $b(k,p,q)$ given by

$$a(k,p,q) = P_{\alpha\beta\gamma}(\mathbf{k})\, P_{\beta\delta}(\mathbf{p})\, P_{\gamma\rho}(\mathbf{q})\, P_{\alpha\delta\rho}(\mathbf{k})/(4k^2), \tag{2.17}$$

$$b(k,p,q) = -P_{\alpha\beta\gamma}(\mathbf{k})\, P_{\beta\delta}(\mathbf{q})\, P_{\gamma\delta\alpha}(\mathbf{p})/(2k^2). \tag{2.18}$$

In (2.16)–(2.18) it is understood that $\mathbf{q} = -\mathbf{k} - \mathbf{p}$.

The quasi-normal approximation as given by (2.16) is not satisfactory. Some time ago it was shown that unphysical negative values for $U(k,t)$ $[E(k,t)]$ are obtained in evolution after a finite time (Ogura 1963). However, it is important to understand better the cause of this difficulty and others which have not previously been brought out in the literature. In the next section we present such a critical discussion of the quasi-normal theory. This is not done with the purpose of making a 'scapegoat', but rather with the goal of giving the reader a glimpse of the many insights into the closure problem that the quasi-normal theory, through its failure, has provided. In this way, we shall be led in § 4 to a modification of (2.16) which gives sensible behaviour.

3. Dynamical properties of the quasi-normal approximation

(i) *Reversibility*

Insight into the nature of the quasi-normal approximation is most readily gained by applying the approximation to model problems which isolate essential features of the Navier–Stokes equations. It must be admitted that results obtained in this way may not bear directly on the approximation applied to Navier–Stokes turbulence. However, we shall not be so conservative in extrapolating from the model dynamics to the full problem.

The essential purpose of the models introduced here is to isolate the effect of an infinite number of degrees of freedom in (2.3) (due to the continuity of the spatial variables $\mathbf{x}$) from the effects of phase mixing of a finite number of degrees of freedom (due to averaging over the ensemble). We begin by considering the inviscid truncated Navier–Stokes equation,

$$\frac{\partial}{\partial t} u_\alpha(\mathbf{k},t) = -\frac{i}{2} P_{\alpha\beta\gamma}(\mathbf{k}) \sum_{\substack{|\mathbf{p}|\leqslant K \\ |\mathbf{k}-\mathbf{p}|\leqslant K}} u_\beta(\mathbf{p},t)\, u_\gamma(\mathbf{k}-\mathbf{p},t), \quad |\mathbf{k}| \leqslant K, \tag{3.1}$$

$$k_\alpha u_\alpha(\mathbf{k},t) = 0, \quad |\mathbf{k}| \leqslant K, \tag{3.2}$$

in which all modal interactions are neglected which involve wave-vectors

above the cut-off K. When L (periodicity length) and K are finite the number of retained modes in (3.1) is finite.

Equation (3.1) conserves energy, namely

$$\frac{\partial}{\partial t}\sum_{|\mathbf{k}|\leqslant K} u_\alpha(\mathbf{k},t)\,u_\alpha(-\mathbf{k},t) = 0. \tag{3.3}$$

In contrast, the zero-viscosity limit of (2.3) is singular and a finite rate of energy dissipation persists in the limit $\nu \to 0$ (Batchelor 1953, p. 103). Maintenance of a non-zero rate of energy dissipation as $\nu \to 0$ (with fixed turbulent energy density) requires that the dissipation occur in shear layers whose thickness approaches zero in the limit. Since only wave-numbers less than K are retained in (3.1), shear layers are spread over distances of order $1/K$ and the truncated system gives zero energy dissipation in the zero-viscosity limit.

The motion given by (3.1) may be studied by classical statistical mechanics. Introduce a phase space whose co-ordinate axes are the independent components of $\mathbf{u}(\mathbf{k})$, $|\mathbf{k}| \leqslant K$. A particular realization of (3.1) is represented, at each instant of time, by a point in this phase space. An ensemble of realizations at $t = 0$ is represented by a distribution of points. As time advances the evolution of the ensemble redistributes the points in a way represented schematically in figure 1. The motion of the ensemble conserves the volume of tiny elements of phase, a result that is expressed mathematically by Liouville's theorem,

$$\sum_{|\mathbf{k}|\leqslant K} \frac{\partial[\partial u_\alpha(\mathbf{k},t)/\partial t]}{\partial u_\alpha(\mathbf{k},t)} = 0, \tag{3.4}$$

which follows directly from (3.1).

If the energy (3.3) is the only *isolating* integral of motion, then the system (3.1) will be ergodic on constant-energy surfaces. This appears to be a reasonable assumption for (3.1) in three dimensions. (In two dimensions, there are additional isolating integrals corresponding to vorticity conservation.) Ergodicity† implies that an arbitrary smooth initial ensemble confined to a constant-energy surface converges weakly to a uniform distribution over the energy surface. Weak convergence means essentially that finite-order moments of the distribution converge. The distribution does not itself converge to uniform values everywhere (which would violate Liouville's theorem that density in phase is conserved), but rather develops much fine-grained structure (as in figure 1(*b*)). When fine-grained structure is integrated in forming a moment, the distribution gives values indistinguishable from a uniform distribution.

The uniform distribution of phase over surfaces of constant energy satisfies

$$\langle u_\alpha(\mathbf{k},t)\,u_\beta(-\mathbf{k},t)\rangle = CP_{\alpha\beta}(\mathbf{k}), \tag{3.5}$$

$$\langle u_\alpha(\mathbf{k},t)\,u_\beta(\mathbf{p},t)\,u_\gamma(-\mathbf{k}-\mathbf{p},t)\rangle = 0, \tag{3.6}$$

where C is a constant. (In fact, if $KL \gg 1$, the number of retained modes is large

† The precise property required is mixing, which is somewhat stronger than ergodicity. We assume that the system (3.1) is mixing as well as ergodic.

and it may be shown that moments of a distribution which is a function of energy alone have values indistinguishable from those of a Gaussian distribution. For this latter distribution all cumulants of order higher than second vanish. This result is not necessary to derive (3.5), (3.6).)

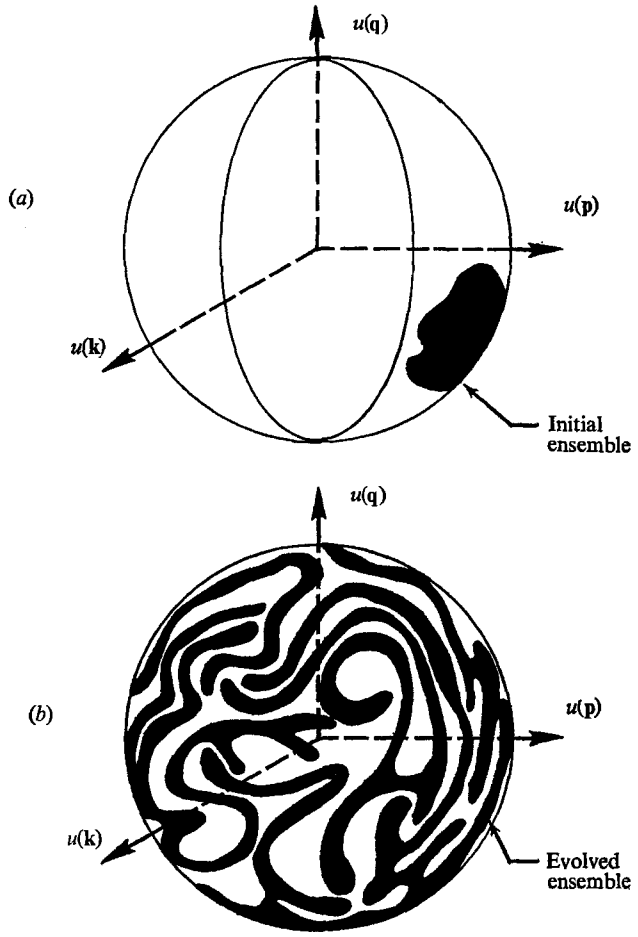

FIGURE 1. Phase space projected along the directions $u(\mathbf{k})$, $u(\mathbf{p})$, $u(\mathbf{q})$ with constant-energy surface indicated: (*a*) representative initial ensemble; (*b*) same ensemble later in its evolution.

In summary, ergodicity on surfaces of constant energy and Liouville's theorem imply that the second- and third-order cumulants of arbitrary smooth initial ensembles should approach as $t \to \infty$ the equilibrium values (3.5), (3.6) with C determined by the total energy of the system. This property should be expected for ensembles consisting of a continuous distribution of phase; ensembles with a finite number of members do not have such behaviour.

The quasi-normal theory may easily be reformulated for the system (3.1).

The result is

$$\frac{\partial}{\partial t}\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle = -\frac{i}{2}\sum_{\substack{|\mathbf{p}|\leqslant K\\ |\mathbf{k}-\mathbf{p}|\leqslant K}}[P_{\alpha\rho\sigma}(\mathbf{k})\langle u_\rho(\mathbf{p},t)u_\sigma(\mathbf{k}-\mathbf{p},t)u_\beta(-\mathbf{k},t)\rangle + P_{\beta\rho\sigma}(-\mathbf{k})\langle u_\alpha(\mathbf{k},t)u_\rho(-\mathbf{p},t)u_\sigma(\mathbf{p}-\mathbf{k},t)\rangle], \quad (3.7)$$

$$(\partial/\partial t)\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{p},t)u_\gamma(\mathbf{q},t)\rangle = -i[P_{\alpha\rho\sigma}(\mathbf{k})\langle u_\beta(\mathbf{p},t)u_\rho(-\mathbf{p},t)\rangle\langle u_\gamma(\mathbf{q},t)u_\sigma(-\mathbf{q},t)\rangle + P_{\beta\rho\sigma}(\mathbf{p})\langle u_\alpha(\mathbf{k},t)u_\rho(-\mathbf{k},t)\rangle\langle u_\gamma(\mathbf{q},t)u_\sigma(-\mathbf{q},t)\rangle + P_{\gamma\rho\sigma}(\mathbf{q})\langle u_\alpha(\mathbf{k},t)u_\rho(-\mathbf{k},t)\rangle\langle u_\beta(\mathbf{p},t)u_\sigma(-\mathbf{p},t)\rangle]. \quad (3.8)$$

It will be shown that the system (3.7), (3.8) is not consistent with approach to the equilibrium values (3.5), (3.6) and the notion of ensemble average.

Suppose that the system (3.7), (3.8) is solved with non-equilibrium initial values for second- and third-order cumulants at $t = 0$. If the cumulants tend to the values (3.5), (3.6), there will exist for arbitrary $\epsilon > 0$ a time t_0 when

$$|\langle u_\alpha(\mathbf{k},t_0)u_\beta(\mathbf{p},t_0)u_\gamma(\mathbf{q},t_0)\rangle| < \epsilon,$$

for all retained $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$, α, β, γ. At $t = t_0$, apply the *small* perturbation (ϵ small),

$$\langle u_\alpha(\mathbf{k},t_0)u_\beta(-\mathbf{k},t_0)\rangle \to +\langle u_\alpha(\mathbf{k},t_0)u_\beta(-\mathbf{k},t_0)\rangle,$$
$$\langle u_\alpha(\mathbf{k},t_0)u_\beta(\mathbf{p},t_0)u_\gamma(\mathbf{q},t_0)\rangle \to -\langle u_\alpha(\mathbf{k},t_0)u_\beta(\mathbf{p},t_0)u_\gamma(\mathbf{q},t_0)\rangle. \quad (3.9)$$

Since (3.7), (3.8) are time-reversible (changing the direction of time and the sign of third-order cumulants keeps the equations invariant), the perturbed system evolves from t_0 to $2t_0$ as the mirror image of the evolution of the unperturbed system from 0 to t_0. In this way, the system unwinds, so that at $t = 2t_0$ the perturbed second-order cumulant equals its initial unperturbed value while the perturbed third-order cumulant equals the negative of its initial unperturbed value. We conclude that the system (3.7), (3.8) cannot approach the equilibrium values (3.5), (3.6) in a stable way. But the approach to equilibrium should be such that ensemble averages are stable to small perturbations—indeed, the purpose of the ensemble is to average over fine details and give smooth coarse-grained results. Equations for ensemble-averaged quantities should not be subject to instabilities and turbulence!

Some comments are in order concerning this reversibility argument. First, the perturbation (3.9) is justifiably small because the number of perturbed quantities is finite. Secondly, the given argument may be generalized to the limit $L \to \infty$, K finite, in which there is a continuum of excited degrees of freedom. Thirdly, the argument may be generalized to the full sequence of closures obtained by neglecting the effects of cumulants above some order. In fact, the full unclosed hierarchy of §2 is formally time-reversible under change of sign of all odd-order cumulants. This latter statement seems to contradict universal approach to the equilibrium values (3.5), (3.6). However, in order to reverse evolution of the *unclosed* hierarchy it is necessary to change the sign of an *infinite* number of high-order cumulants. The point is that such a change of sign is not a small perturbation (since the values of the cumulants do not approach

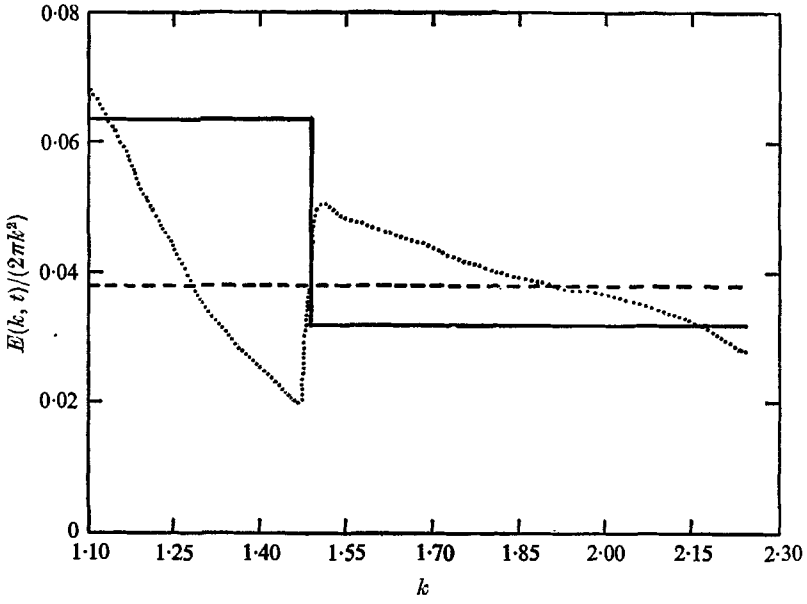

FIGURE 2. Evolution of the energy spectrum according to the quasi-normal equations (2.14), (3.7), (3.8) with wave-vector space truncated to $2^{\frac{1}{8}} = 1{\cdot}09051 \leqslant |\mathbf{k}| \leqslant 2^{47/40} = 2{\cdot}25793$. ——, spectrum at $t = 0$; ....., spectrum at $t = 20$ given by quasi-normal equations; – – –, equipartition spectrum with total energy equal to that of the initial spectrum.

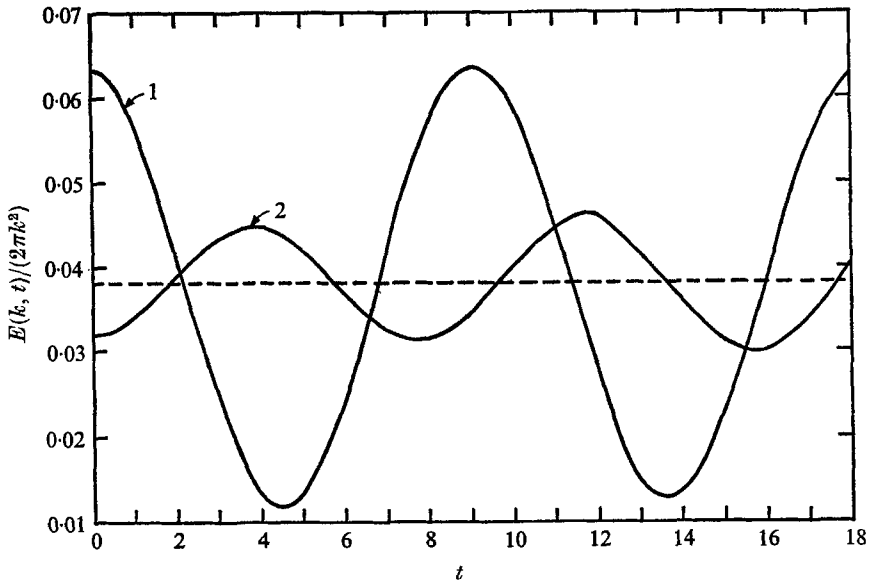

FIGURE 3. Curve 1, the evolution in time of the energy density in wave-number $k = 2^{\frac{3}{10}} = 1{\cdot}23114$ for the quasi-normal calculation described in the caption of figure 1; curve 2, the evolution of the energy density in $k = 2^{\frac{7}{10}} = 1{\cdot}62448$; – – –, the equilibrium density.

zero uniformly). There is nothing inconsistent between formal time-reversibility of the hierarchy and irreversible relaxation to equilibrium. On the other hand, *time reversibility of a closure and irreversible relaxation are inconsistent.*

The conclusion to be drawn from the reversibility argument is that irreversible relaxation of the system (3.1) to equilibrium is misrepresented by any finite-order cumulant-discard closure of the hierarchy. Several other classes of closure including Wiener–Hermite expansions have similar difficulties associated with them (Orszag & Bissonnette 1967). Evidently the physics of the closure problem is intimately bound up with irreversible relaxation effects. These effects are the subject of detailed examination in a forthcoming monograph by the author.

Equations (3.7), (3.8) have been integrated numerically in order to illustrate how the quasi-normal approximation evades approach to equilibrium and misrepresents relaxation effects. The results of some of these calculations are shown in figures 2, 3. The phase coherence observed in these figures is unexplained.

(ii) *Relaxation time*

Further insight into the nature of the quasi-normal approximation's misrepresentation of relaxation effects is obtained by considering a conservative weakly driven model system. The model is constructed to maintain the energy integral (3.3). The model equation is

$$\left[\frac{\partial}{\partial t}+\nu(k)\right]u_\alpha(\mathbf{k},t) = -\frac{i}{2}P_{\alpha\beta\gamma}(\mathbf{k})\sum_{\substack{|\mathbf{p}|\leqslant K\\ |\mathbf{k}-\mathbf{p}|\leqslant K}} u_\beta(\mathbf{p},t)\,u_\gamma(\mathbf{k}-\mathbf{p},t), \quad |\mathbf{k}|\leqslant K, \tag{3.10}$$

with

$$\nu(k) = \begin{cases} -\sigma\sum\limits_{p\in B_2} u_\alpha(\mathbf{p},t)u_\alpha(-\mathbf{p},t) & (k\in B_1),\\ +\sigma\sum\limits_{p\in B_1} u_\alpha(\mathbf{p},t)u_\alpha(-\mathbf{p},t) & (k\in B_2),\\ 0 & \text{otherwise,}\end{cases} \tag{3.11}$$

where B_1, B_2 are non-overlapping wave-number bands. The model (3.10), (3.11) satisfies (3.3) but violates Liouville's theorem (3.4), so that the equilibrium values (3.5), (3.6) are no longer stationary. However, if σ is small, weak driving in band B_1, weak damping in band B_2, and overall conservation indicates that an initial equilibrium ensemble should relax to a state with a slight excitation gradient from B_2 to B_1.

The quasi-normal theory for the model (3.10), (3.11) is obtained by replacing the left-hand sides of (3.7), (3.8) by

$$\left[\frac{\partial}{\partial t}+2\langle\nu(k)\rangle\right]\langle u_\alpha(\mathbf{k},t)u_\beta(-\mathbf{k},t)\rangle,$$

$$\left[\frac{\partial}{\partial t}+\langle\nu(k)\rangle+\langle\nu(p)\rangle+\langle\nu(q)\rangle\right]\langle u_\alpha(\mathbf{k},t)u_\beta(\mathbf{p},t)u_\gamma(\mathbf{q},t)\rangle,$$

respectively. In figure 4 we plot the result of numerical integration of these quasi-normal equations under conditions indicated in the figure caption. It is evident that the system overshoots its new stationary state and does not relax. The trouble here is qualitative and very basic: *Relaxation times for small departures*

from equilibrium should be determined by the random motions in the equilibrium state, not by the departure from equilibrium. In the quasi-normal theory there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached.

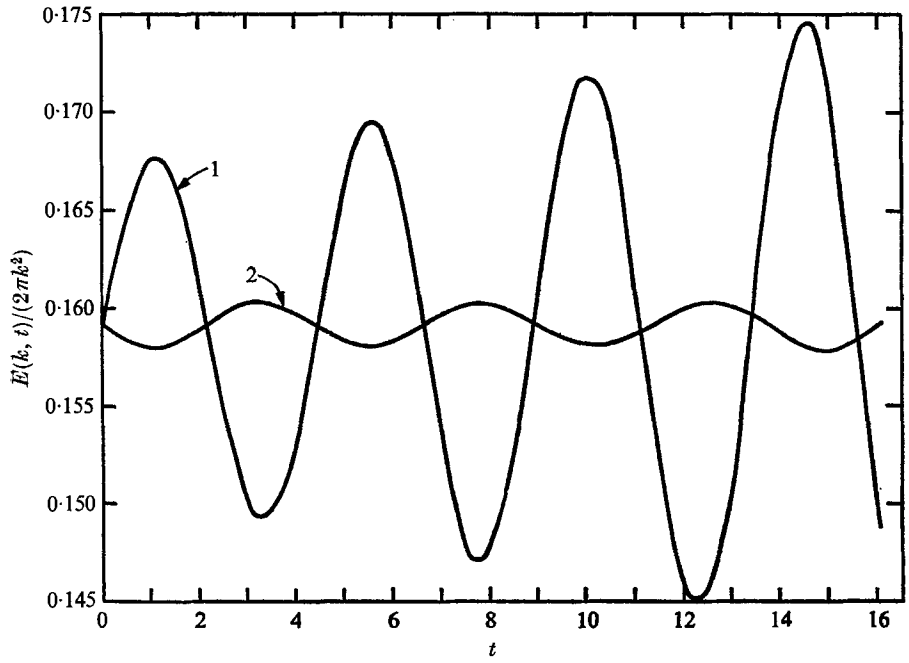

FIGURE 4. Evolution of the energy density in wave-number $k = 2^{\frac{3}{20}} = 1{\cdot}10957$ (curve 1) and $k = 2^{\frac{23}{20}} = 2{\cdot}21914$ (curve 2) using the quasi-normal equations for the weakly driven model equation (3.10), (3.11) with $\sigma = 0{\cdot}05$ and $E(k, 0) = k^2$. Here the band B_1 is $2^{\frac{1}{8}} = 1{\cdot}09051 \leqslant |\mathbf{k}| \leqslant 2^{\frac{9}{40}} = 1{\cdot}16878$, the band B_2 is $2^{\frac{43}{40}} = 2{\cdot}10672 \leqslant |\mathbf{k}| \leqslant 2^{\frac{47}{40}} = 2{\cdot}25793$, and wave-vector space is truncated to $1{\cdot}09051 \leqslant |\mathbf{k}| \leqslant 2{\cdot}25793$.

The model discussed in §3(i) shows that quasi-normal-type closures do not relax to equilibrium; the present model indicates they do not describe properly the motions in the equilibrium state.

(iii) *Realizability*

A third important defect of the quasi-normal approximation concerns unphysical consequences of (2.16) when it is applied to initial-value problems for the decay of isotropic turbulence. Since $u_\alpha(\mathbf{k}, t) = [u_\alpha(-\mathbf{k}, t)]^*$ for a real velocity field, it follows from (2.14), (2.15) that

$$E(k,t) = \lim_{L\to\infty} 2\pi k^2 \left(\frac{L}{2\pi}\right)^3 \langle u_\alpha(\mathbf{k},t) u_\alpha(-\mathbf{k},t)\rangle \geqslant 0, \tag{3.12}$$

which is an example of a realizability inequality. In numerical integrations of (2.16) at moderate Reynolds number, Ogura (1963) found significant violation of this inequality.

In figures 5–8 we show the results of integrations similar to Ogura's. In these figures, the quantities

$$\tfrac{3}{2}[v_{\mathrm{rms}}(t)]^2 = \int_0^\infty E(k,t)\,dk, \tag{3.13}$$

$$\lambda(t) = \left[5\int_0^\infty E(k,t)\,dk \Big/ \int_0^\infty k^2 E(k,t)\,dk\right]^{\frac{1}{2}}, \tag{3.14}$$

$$R_\lambda(t) = v_{\mathrm{rms}}(t)\,\lambda(t)/\nu \tag{3.15}$$

characterize the initial conditions. $\lambda(t)$ is called the Taylor microscale and $R_\lambda(t)$ is the Reynolds number based on this scale. $R_\lambda \sim 20$–50 is typical of a low-turbulence wind tunnel, while $R_\lambda \sim 3000$ is the highest Reynolds number at which accurate measurements of the inertial range have been made (Grant, Stewart & Moilliet 1962).

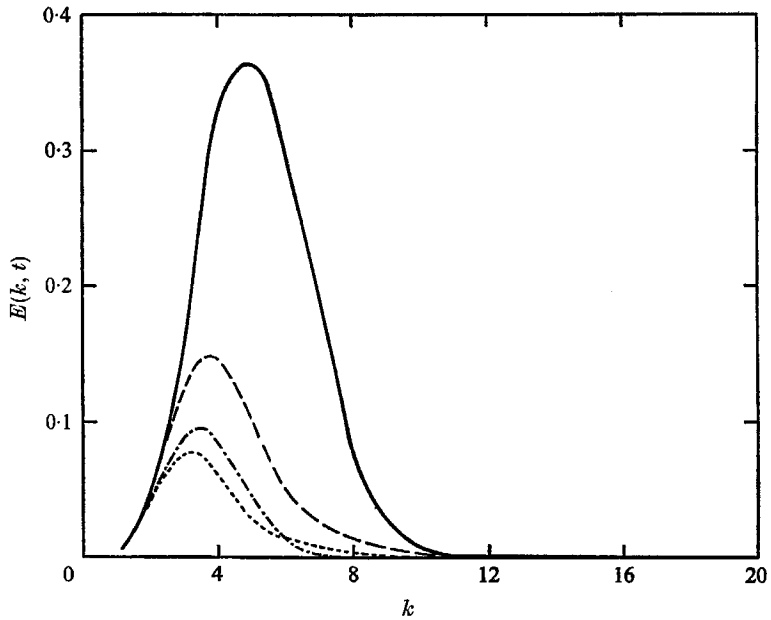

FIGURE 5. Decay calculation using the quasi-normal equation (2.16) with $\nu = 0{\cdot}08$ and (3.16) for the initial energy spectrum. The choice of initial spectrum gives $v_{\mathrm{rms}}(0) = 1$. Wave-vector space was truncated into quarter-octave bands between $k_{\mathrm{bot}} = 2^{\frac{1}{8}} = 1{\cdot}09051$ and $k_{\mathrm{top}} = 2^{\frac{43}{8}} = 41{\cdot}4990$ and the time step was $\Delta t = 0{\cdot}0025$. These conditions give the initial Reynolds number $R_\lambda(0) = 5{\cdot}25$, and Taylor microscale $\lambda(0) = 0{\cdot}4202$. $L_p(0) = 0{\cdot}5248$ is the longitudinal integral scale. The curve marked pure viscous decay is computed using (2.13). ——, $t = 0$; – – –, $t = 0{\cdot}50L_p(0)/v_{\mathrm{rms}}(0)$; ..., $t = 1{\cdot}00L_p/v_{\mathrm{rms}}(0)$; —·—·—, pure viscous decay, $t = 1{\cdot}00L_p(0)/v_{\mathrm{rms}}(0)$.

The initial energy spectrum in figures 5–8 is chosen to be

$$E(k,0) = 16(2/\pi)^{\frac{1}{2}} k_{\max}^{-5} k^4 \exp\left[-2(k/k_{\max})^2\right], \tag{3.16}$$

where $k_{\max} = 4 \times 2^{\frac{1}{4}} = 4{\cdot}75683$ is the wave-number of maximum initial excitation. The spectrum (3.16) gives $v_{\mathrm{rms}}(0) = 1$. At $R_\lambda \sim 5$ (figure 5), no unphysical

behaviour is observed. In fact, there appears little difference between the solution of (2.16) and the pure viscous decay solution (2.13). At higher Reynolds numbers (figures 6, 7) it is seen that the energy density in modes that are strongly excited initially becomes negative after about one circulation time of the energy-containing eddies. In figure 8 the evolution at $R_\lambda \sim 42$ of the energy density in the mode with the highest initial excitation is plotted as a function of time to show that it goes smoothly through zero.

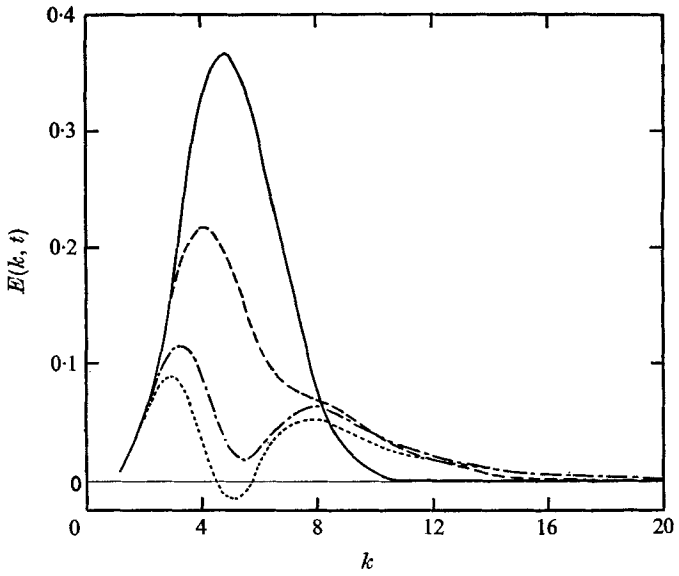

FIGURE 6. Decay calculation using the quasi-normal equation with $\nu = 0{\cdot}02$ and $E(k, 0)$, $v_{\mathrm{rms}}(0)$, $k_{\max}$, $\lambda(0)$, $L_p(0)$, k_{bot}, k_{top}, Δt identical with that given in the caption of figure 5. The initial Reynolds number is $R_\lambda(0) = 21$. ——, $t = 0$; – – –, $t = 0{\cdot}50L_p(0)/v_{\mathrm{rms}}(0)$; —·—·—, $t = 1{\cdot}00L_p(0)/v_{\mathrm{rms}}(0)$; ..., $t = 1{\cdot}27L_p(0)/v_{\mathrm{rms}}(0)$.

This unphysical behaviour may also be understood on the basis of improper relaxation times. At large Reynolds number, the exponential factor in (2.16) is close to unity for energy-containing wave-numbers and times not too large. Therefore, the integral in (2.16) taken over the past history of the turbulence weights all instants from 0 to t approximately equally in computing the contribution to $\partial U(k,t)/\partial t$. If the quasi-normal approximation has plausible small-time behaviour—and it does because it agrees with the first four terms of a formally exact expansion in powers of t about a Gaussian initial distribution—then (2.16) may be expected to tend initially towards a proper quasi-equilibrium decay state of the turbulence. Even if (2.16) were to predict evolution to a quasi-equilibrium state the significant weighting of instants s when $U(k, s)$ is far from this state would imply that $U(k, t)$ would overshoot and not immediately relax. Consistent with this argument is the result that wave-numbers with highest initial excitation, needing the most internal reorganization to reach a quasi-equilibrium state, overshoot most and evolve to negative energy densities.

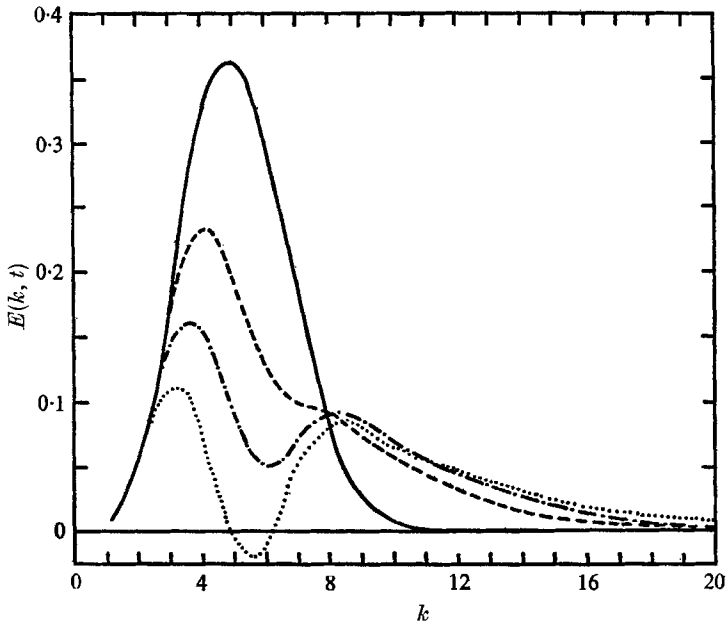

FIGURE 7. Decay calculation using the quasi-normal equation with $\nu = 0{\cdot}01$ and $E(k, 0)$, $v_{\mathrm{rms}}(0)$, $k_{\max}$, $\lambda(0)$, $L_p(0)$, k_{bot}, k_{top}, Δt identical with that given in the caption of figure 5. The initial Reynolds number is $R_\lambda(0) = 42$. ——, $t = 0$; – – –, $t = 0{\cdot}50L_p(0)/v_{\mathrm{rms}}(0)$; —·—·—, $t = 0{\cdot}75L_p(0)/v_{\mathrm{rms}}(0)$; ..., $t = 1{\cdot}00L_p(0)/v_{\mathrm{rms}}(0)$.

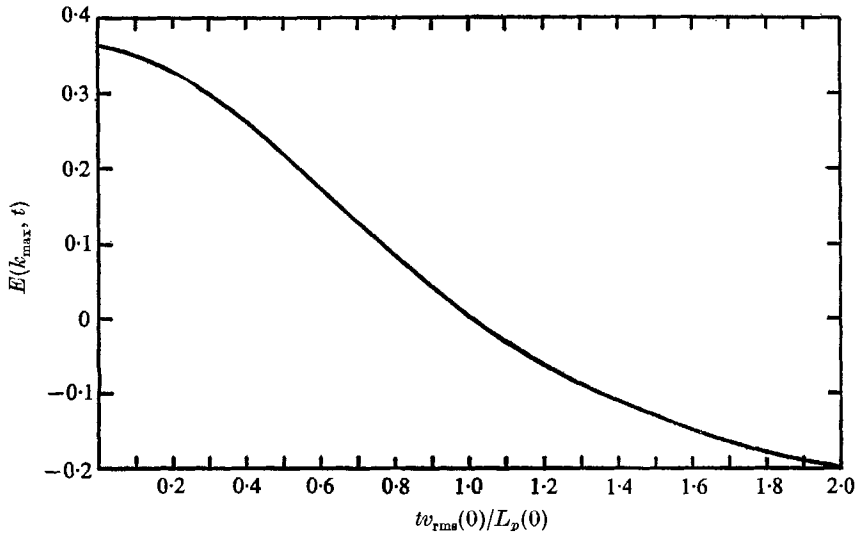

FIGURE 8. Evolution of $E(k_{\max}, t)$ for the calculation shown in figure 7. Here $k_{\max} = 4 \times 2^{\frac{1}{4}}$ and $R_\lambda(0) = 42$.

At low Reynolds numbers, the exponential factor in (2.16) is cut off by viscous dissipation, dynamical memory is finite, and plausible behaviour results.

It is important to emphasize that violation of realizability by (2.16) is not in itself disqualifying. The quasi-normal approximation is, after all, an approximation and, if the negative energy densities only appeared in weakly excited uninteresting regions of Fourier space, the theory might still give a useful approximation to the gross features of turbulence. Violation of realizability by the quasi-normal approximation *is* disqualifying because the violation occurs strongly in dynamically important regions of Fourier space.

The severe difficulties of (2.16) in the energy-containing range may be made more surprising by noting that experiments indicate close agreement between properties of the energy-containing range and Gaussianity: differences from Gaussian behaviour show up clearly only at small scales. Evidently, the small errors incurred by dropping fourth-order cumulants in the energy-containing range accrue rapidly; paradoxically, it is the presence of fourth-order cumulants that keeps the process close to Gaussian.

We close §3 by mentioning that, while the two difficulties discussed in (i) and (ii) applied to all cumulant-discard closures, it is not known whether these higher closures violate energy-density positivity or other realizability inequalities.

4. Eddy relaxation

All three principal difficulties of the quasi-normal approximation discussed in §3 are associated with improper treatment of relaxation effects. The memory integral in (2.16) should not involve only a viscous cut-off when the Reynolds number is large. Instead, the memory should cut off because phase correlations between Fourier modes (or rather eddies) do not persist in the random convection fields of other eddies. Coherence is destroyed by non-linear scrambling.

It is natural to expect improvement on the quasi-normal approximation by taking account of non-linear scrambling with an eddy viscosity in place of molecular viscosity in the memory integral of (2.16); (2.16) is modified to

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]U(k,t) = k^2\int d^3p\int_0^t ds\, e^{-[\eta(k)+\eta(p)+\eta(q)](t-s)} \times [a(k,p,q)\,U(p,s)\,U(q,s)-b(k,p,q)\,U(k,s)\,U(q,s)], \qquad (4.1)$$

where $\eta(k)$ is the eddy viscosity. Methods to derive an equation for the relaxation parameter $\eta(k)$ have been discussed extensively in the literature (Edwards 1964; Kraichnan 1964*b*; Herring 1965, 1966).

In a Kolmogorov inertial range, dimensional analysis gives

$$\eta(k) = a\epsilon^{\frac{1}{3}}k^{\frac{2}{3}},$$

where a is a constant and ϵ is the rate of energy dissipation. With this choice of $\eta(k)$, (4.1) is consistent with a quasi-steady Kolmogorov inertial-range spectrum

$$U(k,t) = b\epsilon^{\frac{2}{3}}k^{-\frac{11}{3}}, \qquad (4.2)$$

where b is a constant.

A more fundamental treatment of $\eta(k)$ is given by the direct-interaction approximation (Kraichnan 1959). This theory involves two new quantities: a two-time correlation function and an average Green's function. The two-time correlation function for isotropic turbulence is defined by

$$U(k;t,t') = \lim_{L\to\infty}\left(\frac{L}{2\pi}\right)^3 \langle u_\alpha(\mathbf{k},t)\, u_\alpha(-\mathbf{k},t)'\rangle, \tag{4.3}$$

so that

$$E(k,t) = 2\pi k^2 U(k;t,t),$$

$$U(k;t,t') = U(k;t',t).$$

The average Green's function $G(k;t,t')$ is defined as the average response in mode k at time t due to a unit impulse perturbation applied to mode k at time t'. The Green's function satisfies $G(k;t,t) = 1$. The direct-interaction equations are

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] U(k;t,t')$$

$$= \tfrac{1}{2}k^2 \int d^3p \left[\int_0^{t'} ds\, a(k,p,q)\, G(k;t',s)\, U(p;t,s)\, U(q;t,s) - \int_0^t ds\, b(k,p,q)\, U(k;t',s)\, G(p;t,s)\, U(q;t,s)\right], \tag{4.4}$$

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] G(k;t,t') = -\tfrac{1}{2}k^2 \int d^3p \int_{t'}^t ds\, b(k,p,q)\, G(k;s,t')\, G(p;t,s)\, U(q;t,s), \tag{4.5}$$

where $a(k,p,q)$, $b(k,p,q)$ are defined by (2.17), (2.18).

It follows from (4.4) that

$$\left[\frac{\partial}{\partial t} + 2\nu k^2\right] U(k,t) = k^2 \int d^3p \int_0^t ds\, [a(k,p,q)\, G(k;t,s)\, U(p;t,s)\, U(q;t,s) - b(k,p,q)\, U(k;t,s)\, G(p;t,s)\, U(q;t,s)]. \tag{4.6}$$

Equation (4.1) may be recovered from (4.6) by the approximations

$$G(k;t,t') = e^{-\eta(k)(t-t')} \quad (t \geqslant t'), \tag{4.7}$$

$$U(k;t,t') = U(k;t',t) = e^{-\eta(k)(t-t')} U(k,t') \quad (t \geqslant t'). \tag{4.8}$$

The relation (4.8), rewritten as

$$U(k;t,t') = G(k;t,t')\, U(k,t') \quad (t \geqslant t'), \tag{4.9}$$

is known to be exact in statistical mechanical equilibrium, where it constitutes the fluctuation-dissipation theorem. However, (4.7) by itself is generally an approximation, even in statistical equilibrium. In non-equilibrium situations including turbulence, (4.9) is also an approximation. In general, the Green's function, not $\eta(k)$, should be considered the fundamental measure of relaxation effects.

5. Properties of the direct-interaction approximation

A less cavalier interpretation of the direct-interaction equations may be given. The direct-interaction equations (4.4)–(4.6) are the *exact* closed set of equations for the two-time correlation function and average Green's function of a model dynamical system bearing some formal similarity to the Navier–Stokes equations. The model equation (Kraichnan 1961) is

$$\left[\frac{\partial}{\partial t}+\nu k^2\right] u_\alpha(\mathbf{k},t) = -\tfrac{1}{2}iP_{\alpha\beta\gamma}(\mathbf{k})\sum_{\mathbf{p}} A(\mathbf{k},\mathbf{p})u_\beta(\mathbf{p},t)u_\gamma(\mathbf{k}-\mathbf{p},t), \tag{5.1}$$

where $A(\mathbf{k},\mathbf{p})$ is a two-valued ($+1$ or -1) zero-mean random variable whose values are independently chosen subject only to the constraint,

$$A(\mathbf{k},\mathbf{p}) = A(\mathbf{p},\mathbf{k}) = A(\mathbf{k},\mathbf{k}-\mathbf{p}) = A(-\mathbf{k},-\mathbf{p}).$$

The latter constraint ensures energy conservation by individual triad interactions and reality of the velocity field.

A cumulant hierarchy similar to that of §2 may be derived from (5.1). However, in contrast to the hierarchy of §2, the new hierarchy may be reduced to (4.4)–(4.6) by the introduction of the two-time correlation function and average Green's function. The new hierarchy is solved exactly using techniques related to those of renormalization in field theory. In this sense, the basic difference between the direct-interaction approximation and cumulant-discard approximations is that the former solves exactly the hierarchy equations for an approximate model system while the latter approximate the solution to the formally exact hierarchy.

The advantage of this derivation of the direct-interaction equations is that several consistency properties follow from it. First, positivity of the energy density must be preserved by (4.4)–(4.6). This may be shown by the same argument used to show (3.12), since the model (5.1) ensures that the physical-space velocity field is real.

Secondly, the existence of equilibrium solutions (3.5), (3.6) follows because the model (with truncated wave-vector interactions) preserves (3.3), (3.4).

Thirdly, approach to equilibrium follows because the model (which should be at least as ergodic as (3.1)) is solved exactly. For turbulence, the model involves an infinity of interacting degrees of freedom, so plausible relaxation properties should be expected.

Numerical integration of the direct-interaction equations (Kraichnan 1964*a*) verifies these properties. In figure 9, approach to equilibrium by the inviscid truncated version of the direct-interaction equations is shown. These calculations (or study of (4.5)) show that $G(k;t,t')$ decays to zero for $|t-t'| \sim (kv_{\rm rms})^{-1}$.

At moderate Reynolds numbers ($R_\lambda \sim 20$–40) numerical integrations of the direct-interaction equations for turbulence decay are in general agreement with experiment. In figure 10, normalized dissipation spectra determined by the direct-interaction equations (curve marked 1) are compared with the wind-tunnel measurements of Stewart & Townsend (1951).

The skewness determined by decay calculations of the direct-interaction

equations is perhaps the most impressive achievement of this theory. The skewness is defined by

$$S(t) = -\langle(\partial v_1(\mathbf{x},t)/\partial x_1)^3\rangle/\langle(\partial v_1(\mathbf{x},t)/\partial x_1)^2\rangle^{\frac{3}{2}}.$$

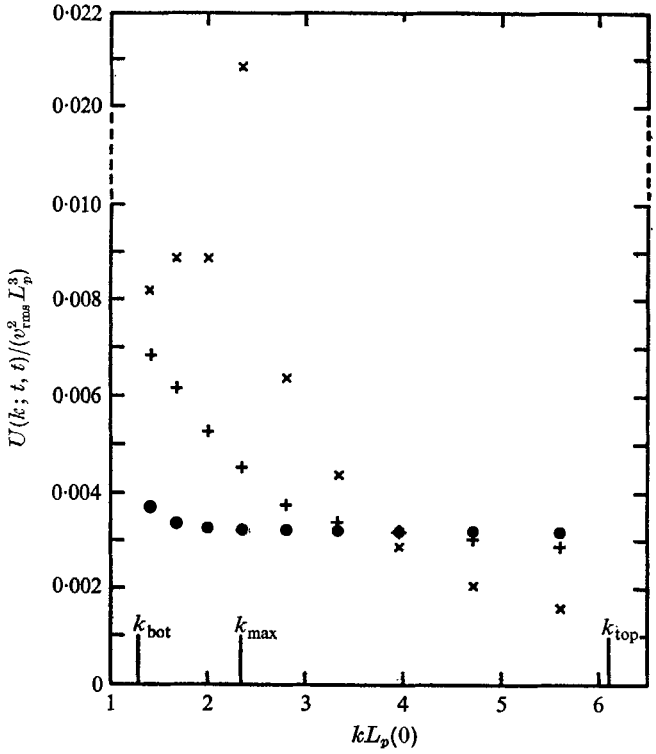

FIGURE 9. Approach to equipartition by the inviscid truncated system using the direct-interaction approximation. The initial spectrum was given by $E(k, 0)$ constant in the quarter-octave band centred on $k_{\max} = 4 \times 2^{\frac{1}{4}}$ and zero elsewhere, with normalization such that $v_{\rm rms}(0) = 1$. The wave-number range was truncated to $2\frac{1}{4}$ octaves, $\Delta t = 0{\cdot}1$ and $L_p(0) = 0{\cdot}4953$. (From Kraichnan 1964*a*.) $tv_{\rm rms}(0)/L_p(0)$: ×, 1·01; +, 2·02; ●, 4·04.

In figure 11 skewnesses determined in a variety of moderate Reynolds number numerical experiments are plotted as a function of time. The value $S \approx 0{\cdot}4$ obtained in this way is consistent with the experimental values for a comparable range of R_λ, as summarized by Batchelor (1953, p. 172). The fit with experiment is obtained without any adjustable parameters in the theory and despite the lack of precise knowledge of experimental initial conditions.

The direct-interaction theory overcorrects the dynamical relaxation of the quasi-normal theory. The memory integrals in (4.6) are cut off too efficiently by non-linear scrambling to be consistent with the Kolmogorov inertial-range spectrum (4.2). As indicated in §4, consistency with the Kolmogorov theory requires $\eta(k) \propto k^{\frac{2}{3}}$, which gives a memory-integral cut-off time of order $k^{-\frac{2}{3}}$. On the other hand, it was mentioned that the Green's function in the direct-interaction theory is affected by non-linear scrambling on the time scale $(kv_{\rm rms})^{-1}$ [$\ll \epsilon^{-\frac{1}{3}}k^{-\frac{2}{3}}$ for k in the inertial range $\epsilon v_{\rm rms}^{-3} \ll k \ll (\epsilon/\nu^3)^{\frac{1}{4}}$].

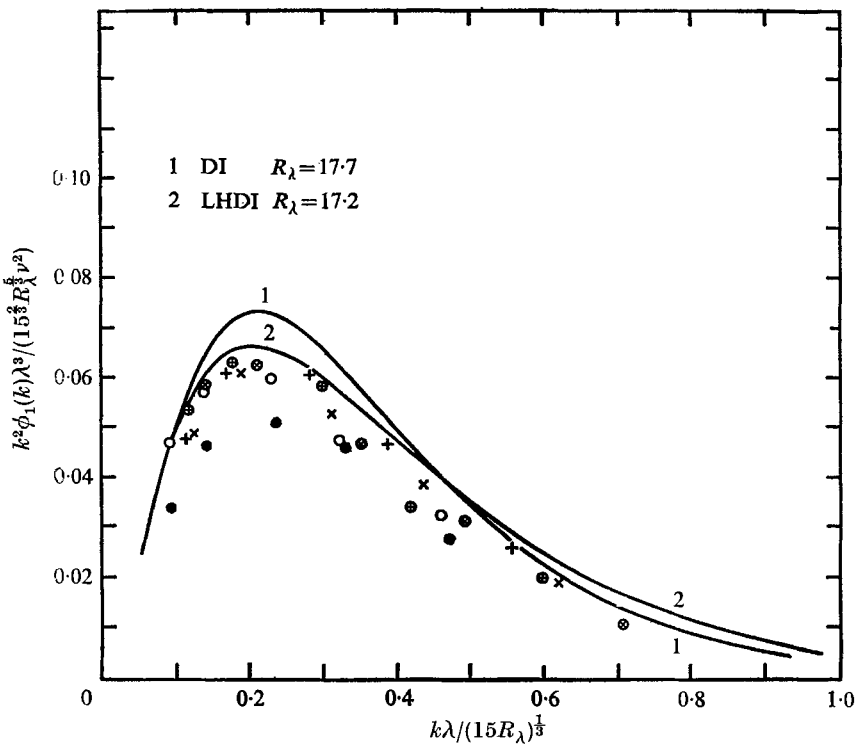

FIGURE 10. Comparison of direct-interaction (curve marked 1) and abridged Lagrangian-history direct-interaction (curve marked 2) decay calculations with data of Stewart & Townsend. The initial spectrum for the calculations was chosen to be

$$E(k, 0) = \tfrac{8}{3}k_{\max}^{-2}k\exp(-k/k_{\max}),$$

with $k_{\max} = 4\times 2^{\frac{1}{4}}$. The choice $\nu = 0{\cdot}01$ gives $R_\lambda(0) = 18{\cdot}98$. The values of R_λ quoted in the figure refer to the spectra after reaching similarity form. The curves show the similarity forms for the one-dimensional dissipation spectra $k^2\phi_1(k)$. The decay spectra quickly evolved to the similarity form. This figure is based on run 10 of Kraichnan (1964*a*). Here

$$\phi_1(k) = \frac{1}{2}\int_k^\infty E(p)\left(1-\frac{k^2}{p^2}\right)\frac{dp}{p}.$$

Stewart & Townsend's data:

	R_λ	x/M		R_λ	x/M
●	14	60	⊙	20	40
+	14	80	⊕	20	60
×	14	100	⊗	20	80

The origin of this defect, which is the most serious deficiency of the direct-interaction theory, is the violation of a statistical form of Galilean invariance. The direct-interaction equations (4.4)–(4.6) are given for a reference frame in which each realization has zero (spatial-)mean velocity. The theory may be reformulated for turbulence in which there is non-zero mean velocity. Galilean invariance holds in the sense that quantities computed in a uniformly moving

co-ordinate system are obtained by Galilean transformation from the zero-mean-velocity co-ordinate system. This invariance is equivalent to the invariance of (5.1) under the transformation

$$\left(\frac{\partial}{\partial t}+\nu k^2\right)\rightarrow\left(\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{U}+\nu k^2\right),$$

$$u_\alpha(\mathbf{k},t)\rightarrow e^{-i(\mathbf{k}\cdot\mathbf{U})t}\,u_\alpha(\mathbf{k},t),$$

where $\mathbf{U}$ is a constant convecting velocity. The trouble with Galilean invariance is more subtle.

If each realization of turbulence is subject to uniform convection whose velocity varies randomly from realization to realization, there should be no observed dynamical effect on simultaneous velocity correlations. On the other hand, random convection with rms velocity V should destroy many-time correlations

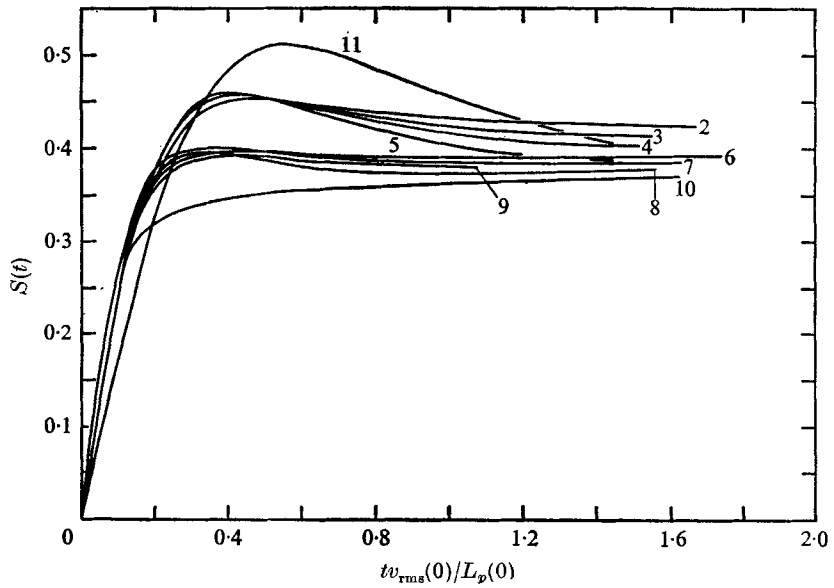

FIGURE 11. Evolution of $S(t)$. The numbering corresponds to the calculations reported by Kraichnan (1964*a*).

such as $U(k;t,t')$, $G(k;t,t')$ on the time scale $|t-t'|\sim(kV)^{-1}$. The convection brings points physically separated by a distance greater than k^{-1}, and hence with little correlation of wave-number k, into geometrical coincidence in the time scale $(kV)^{-1}$. The direct-interaction equations are consistent with this latter behaviour of $U(k;t,t')$ and $G(k;t,t')$. However, since $G(k;t,s)$ and $U(k;t,s)$ $(s\neq t)$ appear in (4.6), $U(k;t,t)$ is affected by the random convection, which it should not be.

This statistical Galilean non-invariance of the direct-interaction equations may be seen from the model (5.1). Because of the random variation of the coupling constants $A(\mathbf{k},\mathbf{p})$ with $\mathbf{k},\mathbf{p}$, excitation at a very low wave-number has effects on high wave-number modes markedly different from excitation at zero wave-number. On the other hand for finite times of evolution the Navier–Stokes

equations show that very low wave-numbers and zero wave-number have approximately the same effect on high wave-numbers, namely uniform convection with weak or zero shear, respectively.

Kraichnan (1965, 1966) has modified the direct-interaction theory to eliminate these spurious relaxation effects of random convection in (4.6). The modification called the Lagrangian-history direct-interaction (LHDI) theory involves reworking (4.4)–(4.6) in Lagrangian co-ordinates. The resulting equation for $U(k,t)$ is

$$\left[\frac{\partial}{\partial t}+2\nu k^2\right]U(k,t) = k^2\int d^3p\int_0^t ds[a(k,p,q)\,G(k;t/s)\,U(p;t/s)\,U(q;t/s) - b(k,p,q)\,U(k;t/s)\,G(p;t/s)\,U(q;t/s)], \quad (5.2)$$

where $G(k;t/t')$ is a Lagrangian Green's function, $U(k;t/t')$ is a Lagrangian correlation function, and $a(k,p,q)$, $b(k,p,q)$ are still given by (2.17), (2.18). The LHDI theory gives equations for $G(k;t/t')$ and $U(k;t/t')$ such that the simultaneous correlation $U(k,t)$ is not affected by random convection.

The LHDI theory is consistent with the Kolmogorov theory. It gives an inertial-range energy spectrum

$$E(k,t) = C\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}},$$

with $C = 1{\cdot}77$. In figure 12, we show the impressive fit of the spectrum given by the LHDI theory (Kraichnan 1966) with the high-Reynolds-number ($R_\lambda \sim 3000$) tidal-channel measurements of Grant *et al.* (1962). It is impressive that predicted and measured dissipation spectra both peak at a wave-number $0{\cdot}09(\epsilon/\nu^3)^{\frac{1}{4}}$.

In figure 10, the curve marked 2 shows that at moderate Reynolds numbers the Lagrangian theory gives results consistent with both experiment and the unmodified direct-interaction theory.

It has not yet been possible to derive the Lagrangian theory as convincingly as the direct-interaction theory was given by the model (5.1). At present, the Lagrangian modifications remain a heuristically motivated set of rules without fundamental justification. However, the theory's successful application to homogeneous turbulence suggests that a more fundamental derivation should be possible.

6. Numerical calculation of the energy spectrum of homogeneous turbulence

In §6, we compare two methods for obtaining information about the energy spectrum of homogeneous turbulence: numerical solution of the Navier–Stokes equations and numerical solution of the direct-interaction equations.

It seems that the most efficient numerical simulation of the Navier–Stokes equations with N mesh points involves (Orszag 1969) $(a\log_2 N+b)N$ arithmetic operations† per time step, where a is of order 1 and b is of order 10. The factor $aN\log_2 N$ is due to the solution of the Poisson equation for the pressure field in physical space

$$\nabla^2 p = -\boldsymbol{\nabla}.[\mathbf{v}.\boldsymbol{\nabla})\,\mathbf{v}],$$

† An operation is defined to be a multiplication followed by an addition.

or the evaluation of the convolution sums in (2.3) in Fourier space. The number of mesh points in physical space must be chosen to resolve the finest scale of appreciable excitation, namely shear layers of thickness the Kolmogorov dissipation scale. In Fourier space, modes below the Kolmogorov dissipation wave-number must be retained. Both these estimates give

$$N \gtrsim (k_d L_p)^3 = O(R_\lambda^{\frac{9}{2}}),$$

where $k_d = (\epsilon/\nu^3)^{\frac{1}{4}}$ is the Kolmogorov dissipation wave-number and L_p is the longitudinal integral scale.

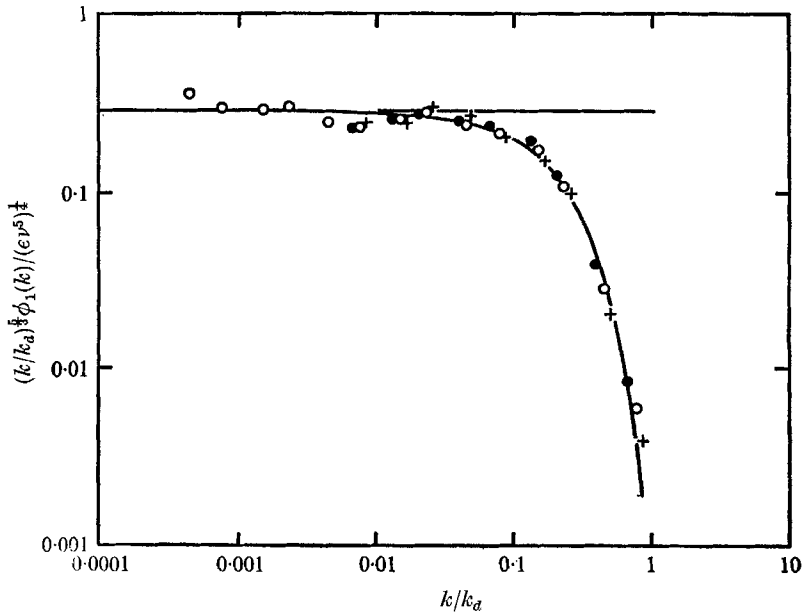

FIGURE 12. Computed non-dimensionalized spectrum function $k^{\frac{5}{3}}\phi_1(k)$ in the inertial and dissipation ranges compared with the October 1959 data of Grant, Stewart & Moilliet. The dissipation wave-number k_d is defined by $k_d = (\epsilon/\nu^3)^{\frac{1}{4}}$, where ϵ is the rate of energy dissipation. (From Kraichnan 1966.)

The time step must be chosen smaller than the time it takes an eddy of size k_d^{-1} to be swept over its diameter by the large-scale field and smaller than the viscous diffusion time $(\nu k_d^2)^{-1}$ over the distance k_d^{-1}. The former, which is the more stringent requirement, gives a time step

$$\Delta t \lesssim R_\lambda^{-\frac{3}{2}} L_p / v_{\text{rms}}.$$

The decay time of turbulence is of the order of the circulation time of the energy-containing eddies which is L_p/v_{rms}. Therefore, the number of time steps necessary to evolve a realization of turbulence through its decay is at least of order $R_\lambda^{\frac{3}{2}}$. Considering the number of operations per time step and the number of time steps, the total number of operations is of order

$$R_\lambda^6 \log_2 R_\lambda. \tag{6.1}$$

At $R_\lambda = 20$, it has been found acceptable to choose $N = (32)^3$ (Orszag 1969). The total number of operations is of order 10^9. With a machine such as the IBM Model 360-91, 95, whose cycle time is less than 1 μsec, this calculation is feasible. Averages are obtained as spatial averages of a single realization. However, the scaling (6.1) makes it apparent that Reynolds numbers significantly higher than this are beyond foreseeable computers. I have estimated that $R_\lambda \sim 3000$ requires 10^{20} operations!

Now consider the solution of the direct-interaction equations (4.4)–(4.6). As in direct numerical simulation, it is necessary to retain wave-numbers up to k_d. However, two facts make this situation more palatable than before. First, $U(k;t,t')$ and $G(k;t,t')$ are functions of the *scalar* wave-number k and, second, $U(k;,t,t')$ and $G(k;t,t')$ are *smooth* functions of k, since they are ensemble averages. Smoothness and the algebraic behaviour of $U(k;t,t')$ in the inertial range indicate that it is not necessary for accuracy of simulation to retain all modes up to k_d but only a fixed number of modes per octave of wave-number. This gives for the number of retained modes

$$N \gtrsim \log_2 k_d L_p = O(\log_2 R_\lambda).$$

The time step must be chosen as in direct simulation to be of order $R_\lambda^{-\frac{3}{2}} L_p / v_{\rm rms}$.

Naïve calculation of (4.4)–(4.6) for M time steps would require order $M^3 N^3$ operations. The factor N^3 comes from the wave-vector integrations and causes no significant problem. The factor M^3 arises from the calculation of $U(k;t,t')$ and $G(k;t,t')$ for M choices each of t,t' and the order M possible choices of s in the memory integrals of (4.4)–(4.6). Since evolution through a decay time involves order $R_\lambda^{\frac{3}{2}}$ time steps, the factor M^3 can make the calculation of the direct-interaction equations little more practical than solution of the Navier–Stokes equations at high Reynolds number.

Fortunately, more sophisticated calculation of the direct-interaction equations reduces the total number of operations to order MN^3. The idea is to calculate only those functions which are appreciably different from zero and only those terms in (4.4)–(4.6) giving non-negligible contribution. For example, $U(k;t,t')$ is negligible when k is in the inertial range and $|t-t'| \gg (kv_{\rm rms})^{-1}$. The result is that the total number of operations necessary to calculate the direct-interaction equations through a decay time is of order

$$R_\lambda^{\frac{3}{2}} (\log_2 R_\lambda)^3. \tag{6.2}$$

At $R_\lambda \sim 40$, Kraichnan (1964*a*) used $N = 20$, $M = 20$. The total number of arithmetic operations required at this Reynolds number is about 10^8 which is only slightly less than is required for solution of the Navier–Stokes equations. However, the advantage of (6.2) over (6.1) increases rapidly with Reynolds number.

7. Conclusions

It must be admitted that the principal result of fifty years of turbulence research is the recognition of the profound difficulties of the subject. This is not meant to imply that a fully satisfactory theory is beyond hope. Rather, the

author sees two promising approaches to the prediction of properties of turbulence. Both depend on the observation that the problem is one of approximation.

First, there is the technique for obtaining convergents to properties of turbulence which is described by Kraichnan elsewhere in these Proceedings. This method provides a systematic way of improving the results given by a theory such as the direct-interaction approximation. The basis for this approach is the observation that most of the formal power series that arise in the calculation of properties of turbulence are divergent. Therefore, poor approximations to the quantity are obtained by truncation of these series. Kraichnan presents a method for extracting useful information from the first few terms of the power series.

Another promising approach is the use of direct numerical simulation of the large-scale features of the turbulence with a theory such as the direct-interaction approximation to represent the sub-grid-scale motions. An interesting first step in this direction, which uses an eddy viscosity to simulate the small scales, is discussed by Deardorff in these Proceedings. At this time, it seems that, if detailed study of intermittent objects of turbulent flows is to be made, this kind of approach is the only practical alternative. The author foresees a successful marriage of sophisticated numerical techniques with turbulence theory to permit solution of problems such as Reynolds' determination of the pressure drop in pipe flow. The next several years will doubtless see rapid developments in this direction.

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